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Mean-field theory and fluctuations in Potts spin glasses: II

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Abstract. The fluctuations in the ordered phase around the mean-field solution are incorporated in a renormalization group approach and it is found that for a small range of the parameters they restore scaling close to the upper critical dimension. For the 3-state case it appears that fluctuations destroy the stability of the solution and cause the system to undergo a first-order phase transition. Non-universal fluctuation corrections to the equation of state above six dimensions are obtained.

1. Introduction

The problem of understanding the effect of fluctuations around the mean-field solution of Ising spin glasses has been a tantalizing one [1-4] and progress has been steady but slow. Many of the difficulties found there are connected to the complexity of the mean-field solution itself. (For a review see [5].)

It is therefore interesting to look at the effect of the fluctuations in spin-glass models which have a simpler structure at the mean-field level, which provides a more accessible starting point.

In a previous paper [6], to be referred to as I, it was shown that an effective Hamiltonian can be obtained for the Potts system with short-range interactions which are randomly distributed, using the replica trick. The replica symmetric solution obtained when looking for a stationary point of the effective Hamiltonian is unstable, and the instability in the 'replicon' fluctuation modes appears at order $t=(T_G-T)/T_G$ (in the Ising case the corresponding modes are marginal at this order and the instability appears at order t^2). A replica symmetry breaking solution (with only one level of replica symmetry breaking) exists, and the corresponding transition is continuous for $p<4$, p being the number of equivalent Potts' states. The Gaussian eigenvalues of fluctuations around this solution are stable for $p>p_0\simeq 2.8$. Two of the modes (the DR mode and the IB mode in the notation introduced in I) are soft in the sense that the correlation length associated with them diverges (at the mean-field level) with an exponent $\nu_{\text{soft}}=1$, which is bigger than the exponent $\nu_{\text{fast}}=\frac{1}{2}$ associated with all the remaining modes, indicating a failure of conventional scaling. The leading term of the mass of the soft modes depends on the quartic coupling y (see (5.2l) of I). The long-range fluctuations in the ordered phase renormalize the quartic coupling already for $d<8$, and they have to be taken into account to obtain the correct temperature dependence and critical exponents near the transition.

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The fluctuations in the direction of the soft modes (in replica space) are the ones that play the dominant role, and when they are computed to one-loop two different behaviours emerge.

The fluctuations associated with the IB family are renormalized and the leading term of the corresponding eigenvalue acquires a $t^{(d-4)/2}$ dependence, which restores the proper scaling behaviour at $d=6$, in the whole range of existence of the mean-field solution, in agreement with Fisher and Sompolinski [7].

The fluctuations associated with the DR replicon modes instead, will only be renormalized in a similar way for $p > p_1 \simeq 3.77$. For $p < p_1$ an instability develops in which the coupling is renormalized away from the 'stability wedge' of the solution discussed in I, a scenario usually associated with a fluctuation-driven first-order phase transition [8, 9].

As the presence of the dangerous quartic coupling requires an understanding of the behaviour of the system above the upper critical dimension (for $6<d<8$), non-universal corrections to the free energy and equation of state are present. To one-loop the effect of those corrections is to lower the p_c (the value of p above which the continuous solution ceases to exist) from its mean-field value $p_c = 4$.

The plan of this paper is as follows. In section 2 we outline the way of evaluating the leading contribution of the fluctuations, in section 3 we apply those results to the renormalization of the intergroup fluctuations, in section 4 we compute the renormalization of the intragroup fluctuations, in section 5 we evaluate corrections to the equation of state, and make some concluding remarks.

2. The leading fluctuations

The cubic terms of the effective Hamiltonian (equation (2.13) of I) can be expressed in terms of the eigenmodes found in appendices 2 and 3 of I, through an orthogonality transformation:

$$\sum_{\alpha\neq\beta} \frac{v_{abc}v_{def}}{p^2} q^{\alpha\beta}_{ad} q^{\alpha\beta}_{be} q^{\alpha\beta}_{cf} \rightarrow \sum_{\mathrm{I}} \sum_{\substack{R_1,R_2,R_3\\P_1,P_2,P_3}} \mathbf{T}^R_A(R_1, R_2, R_3)\mathbf{T}^P_A(P_1, P_2, P_3) q^{\mathrm{I}}_{R_1P_1} q^{\mathrm{I}}_{R_2P_2} q^{\mathrm{I}}_{R_3P_3} + \sum_{\mathrm{I}\neq\mathrm{J}} \sum_{R_1,R_2,R_3} \frac{v_{abc}v_{def}}{p^2} \mathbf{T}^R_A(R_1, R_2, R_3) q^{\mathrm{IJ}}_{R_1,ad} q^{\mathrm{IJ}}_{R_2,be} q^{\mathrm{IJ}}_{R_3,cf} \tag{2.1}$$

$$\sum_{\alpha\neq\beta\neq\gamma\neq\alpha} q^{\alpha\beta}_{ab} q^{\beta\gamma}_{bc} q^{\gamma\alpha}_{ca} \rightarrow \sum_{\mathrm{I}} \sum_{\substack{R_1,R_2,R_3\\P_1,P_2,P_3}} \mathbf{T}^R_B(R_1, R_2, R_3)\mathbf{T}^P_B(P_1, P_2, P_3) q^{\mathrm{I}}_{R_1P_1} q^{\mathrm{I}}_{R_2P_2} q^{\mathrm{I}}_{R_3P_3} + \sum_{\mathrm{I}\neq\mathrm{J}\neq\mathrm{K}} \sum_{R_1,R_2,R_3} \mathbf{T}^R_B(R_1, R_2, R_3) q^{\mathrm{IJ}}_{R_1,ab} q^{\mathrm{JK}}_{R_2,bc} q^{\mathrm{KI}}_{R_3,ca} + 3\sum_{\mathrm{I}\neq\mathrm{J}} \sum_{R_1,R_2,R_3} \sum_{P_1} (P_1)_{ab} \mathbf{T}^R_B(R_1, R_2, R_3) q^{\mathrm{I}}_{R_1P_1} q^{\mathrm{IJ}}_{R_2,bc} q^{\mathrm{IJ}}_{R_3,ac}. \tag{2.2}$$

The notation introduced is as follows. Capital letter indices are group indices, greek letters are replica indices running inside each group and lower case indices are

Potts-component indices. q^{I}_{RP} represents one of the intragroup modes associated with group I; its components $(q^{\mathrm{I}}_{RP})^{\alpha\beta}_{ab}$ will be zero, then, unless α and β belong to the group I; in this case they will be $R_{\alpha\beta}P_{ab}$, with $R_{\alpha\beta}$ the orthogonal combinations of the replica vectors introduced in equations (A2.2) and (A2.3) of I and the Potts vectors P_{ab} being the $(p-1)^2$ vectors of type P_A, P_D, P_V and P_T introduced in appendix 2 of I. $q^{\mathrm{IJ}}_{R,ab}$ is one of the intergroup modes associated with the pair of groups (I, J); its components $(q^{\mathrm{IJ}}_{R,ab})^{\alpha\beta}_{cd}$ are zero when the condition $\alpha \in \mathrm{I}$ and $\beta \in \mathrm{J}$ is not satisfied; otherwise they are $R_{\alpha\beta}\delta_{ac}\delta_{bd}$, with the $R_{\alpha\beta}$ being the orthogonal combinations of the replica vectors introduced in (A3.2) of I.

The replica part couplings are defined

$$\mathbf{T}^R_A(R_1, R_2, R_3) = \sum_{\alpha \neq \beta} (R_1)_{\alpha\beta}(R_2)_{\alpha\beta}(R_3)_{\alpha\beta} \tag{2.3a}$$

$$\mathbf{T}^R_B(R_1, R_2, R_3) = \sum_{\alpha \neq \beta \neq \gamma \neq \alpha} (R_1)_{\alpha\beta}(R_2)_{\beta\gamma}(R_3)_{\gamma\alpha}. \tag{2.3b}$$

The range of the sums over replica indices involved in these definitions has to be understood in each case. For example, the indices α, β, γ in the definition of $\mathbf{T}^R_B$ run over the group I in the first term of expression (2.2); they ran over the groups I, J and K, respectively, in the second term of that expression; and α and β run over group I and γ over group J in the last term.

The general expressions for these cubic replica couplings are very complicated but they can be obtained easily in some particular cases in which we will need them. The Potts indices' couplings are defined as

$$\begin{aligned} \mathbf{T}^P_A(P_1, P_2, P_3) &= \sum_{\substack{a,b,c\\ d,e,f}} \frac{v_{abc}v_{def}}{p^2}(P_1)_{ad}(P_2)_{be}(P_3)_{cf} \\ \mathbf{T}^P_B(P_1, P_2, P_3) &= \sum_{a,b,c} (P_1)_{ab}(P_2)_{bc}(P_3)_{ca}. \end{aligned} \tag{2.4}$$

Only one of the terms obtained mixes intergroup and intragroup modes. The first (second) term in the RHS of expressions (2.1) and (2.2) only couples intragroup (intergroup) fluctuations among themselves.

An important simplification in these expressions can be achieved by the following observation. As we will argue below, in the range $6 < d < 8$ the only coupling of the system that has to be renormalized is the quartic coupling, and all the one-loop contributions to its renormalization which are not negligible in the transition region contain integrals of the form

$$\frac{1}{(2\pi)^d}\int \frac{\mathrm{d}^d k}{(k^2+\lambda_1)(k^2+\lambda_2)(k^2+\lambda_3)(k^2+\lambda_4)} \tag{2.5}$$

where the λ_i are eigenvalues associated with the fluctuation modes found previously. These integrals have a divergence (when $t \to 0$) in less than eight dimensions, renormalizing the quartic coupling y away from its original value y_0. Although this renormalization happens in any system with a quartic and cubic term in its Hamiltonian, in general it furnishes only a non-leading correction; in this particular case, because of the cancellation of the leading term in the mass of the soft modes, this renormalization plays a crucial role.

The terms involving *four* soft modes will behave asymptotically as $t^{(d-8)\alpha/2}$, where we assumed that the renormalized eigenmodes are $\lambda_{\mathrm{soft}} \sim t^\alpha$. (Above $d = 8$ α is equal to 2, as found in the mean-field solution, but for $d < 8$ it has to be determined self-consistently from the renormalization calculation). All the other terms where at least one of the modes is not soft, $\lambda_F \sim t$, will be less divergent. (For $d<6$ there are subleading corrections coming from the terms in which exactly one of the modes is fast, which will diverge as an intermediate power.) In the range $6<d<8$ in which we are interested, and close enough to the transition, we need to keep only the contributions of the diagrams where $\lambda_1=\lambda_2=\lambda_3=\lambda_4=$ an eigenvalue associated with a soft mode.

We can use this fact to simplify the cubic terms of the effective Hamiltonian (2.1) and (2.2), neglecting all the terms which will not lead to the most diverging corrections. Two properties of the replica vectors are useful here.

Property P1. All the replicon modes (equations (A2.2*c*), (A2.3*b*) and (A3.2*c*) of I) verify

$$\sum_{\alpha} R_{\alpha\beta} = \sum_{\beta} R_{\alpha\beta} = 0.$$

Property P2. All the anomalous modes (equations (A2.2*b*), (A2.3*a*) and (A3.2*b*) of I) satisfy a weaker condition

$$\sum_{\alpha,\beta} R_{\alpha\beta} = 0.$$

The properties P1 and P2 can be easily verified performing by the sums in the corresponding expressions, and they simply express the orthogonality in replica space of these modes with respect to the breathing mode.

Imposing in expressions (2.1) and (2.2) the condition that at least two of the modes in each term have to be soft modes (as terms that have more than one fast mode will necessarily imply diagrams with one of the internal λ_i being soft and giving a subleading contribution), the cubic terms of the effective Hamiltonian become

$$w\left(\frac{1}{12m}\sum_{\mathrm{I}\neq\mathrm{J}}\frac{v_{abc}v_{def}}{p^2}q^{\mathrm{IJ}}_{ad}q^{\mathrm{IJ}}_{be}q^{\mathrm{IJ}}_{cf}+\frac{1}{6}\sum_{\mathrm{I}\neq\mathrm{J}\neq\mathrm{K}}q^{\mathrm{IJ}}_{ab}q^{\mathrm{JK}}_{bc}q^{\mathrm{KI}}_{ca}+\frac{p-2}{12}\sum_{\mathrm{I}}\mathbf{T}_A(R_1R_2R_3)q^{\mathrm{I}}_{R_1}q^{\mathrm{I}}_{R_2}q^{\mathrm{I}}_{R_3}\right.$$
$$\left.+\frac{1}{6}\sum_{\mathrm{I}}\mathbf{T}_B(R_1,R_2,R_3)q^{\mathrm{I}}_{R_1}q^{\mathrm{I}}_{R_2}q^{\mathrm{I}}_{R_3}+\frac{m-1}{2}\sum_{\substack{\mathrm{I}\neq\mathrm{J}\\ P_1}}(P_1)_{ab}q^{\mathrm{I}}_{P_1}q^{\mathrm{IJ}}_{ac}q^{\mathrm{IJ}}_{bc}\right). \qquad (2.6)$$

In all the terms containing only intergroup modes, by using the properties P1 and P2 in the evaluation of the tensors $\mathbf{T}^R_A$ and $\mathbf{T}^R_B$, only terms coupling soft modes remain (belonging to the IB family). The replica indices are, thus, eliminated. Accordingly q^{IJ}_{ab} stands, from now on, for a *soft* intergroup mode. The structure of these pure intergroup terms (first two terms in expression (2.6)) reproduces the original structure of the cubic couplings of the theory in (2.13) of I, because there is only one intergroup soft mode associated with each pair of replica groups (besides the trivial $(p-1)^2$ Potts degeneracy).

In the case of the pure intragroup terms (the third and fourth terms in expression (2.6)), when the Potts tensors $\mathbf{T}^P_A$ and $\mathbf{T}^P_B$ are evaluated using the fact that the soft modes are diagonal in the Potts indices and that the D modes are the only non-traceless modes in Potts space, only couplings between D modes remain, and the Potts index is no longer necessary. Two of the q^{I}_R in these terms have to be soft modes DR and the third one can be another DR mode or one of the fast DB or DA modes.

In the last term, which mixes both kinds of fluctuations, only the terms where q^{I} is a B mode in replica space survive (using again the properties P1 and P2) and, thus, the modes q^{IJ}_{ab}, q^{IJ}_{cb} must be soft intergroup modes, but the mode q^{I} is fast.

The different vertices obtained are depicted in figure 1, using a modification of a representation first used in this problem by Goldschmidt [10]. Double lines represent intergroup modes (the I families introduced in appendix 3 of I) while single lines represent the intragroup families (D, V, T, A) discussed in appendix 2 of I.

The leading term of the eigenvalues associated with the non-soft modes ($5.2a$–k) of I depends only on the cubic coupling w which is not renormalized for $6 < d < 8$ (it

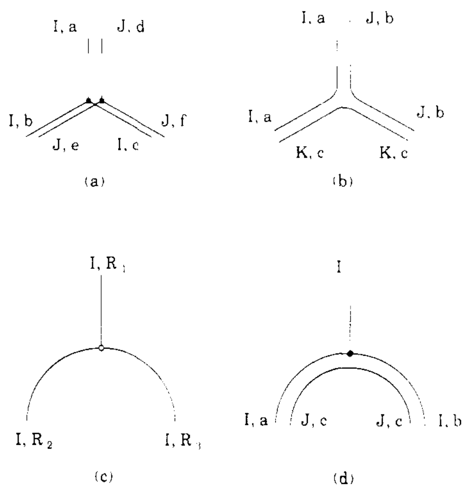

Figure 1. Graphical representation of the cubic vertices responsible for the leading corrections to the quartic coupling. Double lines represent intergroup modes, single lines are intragroup modes. (a) First term in (2.6). The dots represent the v Potts tensors. (b) Second term in (2.6). (c) Third and fourth terms in (2.6). The open dots represent the tensors $\mathbf{T}_A^R$ or $\mathbf{T}_B^R$. (d) Last term in (2.6). The single line represents a mode belonging to the B families, which is non-soft, and should not be connected when obtaining the quartic coupling diagrams.

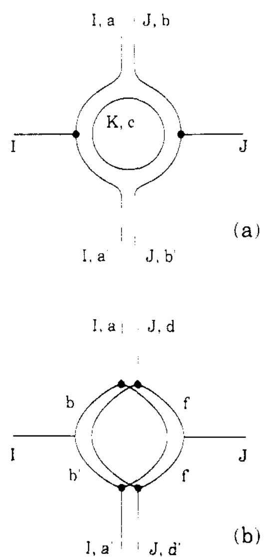

Figure 2. Diagrams contributing to the renormalization of the IB soft eigenvalue.

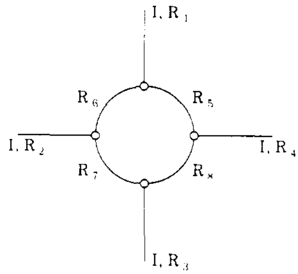

Figure 3. Diagrams contributing to the renormalization of the DR soft eigenvalue.

is an irrelevant variable for $d > 6$), and it is not affected by the renormalization of the quartic coupling. We need only focus, then, on the quartic coupling, which will affect the leading behaviour of the soft modes IB and DR.

The only important diagrams that contribute to its renormalization are represented in figures 2 and 3. The reason for this is discussed in appendix 1. It is interesting to note that to the leading order considered here in the one-loop corrections, the two families of soft modes renormalize independently from each other, i.e. the diagram which renormalizes the intergroup soft mode (IB) in figure 2 only contains internal lines of that type, and the same is true for the intragroup modes (DR) in figure 3. This is important for the self-consistent calculations as the two families behave in different ways under renormalization, as will be discussed later.

3. Renormalization of the intergroup fluctuations

The relevant contributions to the eigenvalues associated with the IB modes coming from the diagrams in figure 2 are

$$c\omega^4 q^2 (m-1)^2 \lambda_{\mathrm{IB}}^{(d-8)/2}\left[\frac{3}{2}\left(\frac{p-2}{m}\right)^2 + 3(p-1)\left(\frac{n}{m}-2\right)\right] q_{\mathrm{IB}}^2 \tag{3.1}$$

where the uninteresting numerical constant c comes from the momentum integral. The derivation is given in appendix 2.

Inserting the mean-field solution values ((4.5) of I), we obtain, close to the transition, in the $n \to 0$ limit

$$-\tfrac{3}{2} c\omega^4 q^2 (p-4)^2 (p-2) \lambda_{\mathrm{IB}}^{(d-8)/2} (1 + \mathrm{O}(yt)) q_{\mathrm{IB}}^2 \tag{3.2}$$

where for internal consistency we require that yt should vanish when $t \to 0$. The sign of this renormalized contribution, which is constant throughout the range of validity of the solution ($p_0 < p < 4$), is a crucial result.

Adding this contribution to the mean-field value of the eigenvalue obtained in (5.2l) of I:

$$\lambda_{\mathrm{IB}} = \frac{t^2}{\omega^2}\left(\frac{y_0(7p^2 - 24p + 12)}{24(4-p)^2} + \tfrac{3}{2} c\omega^4 (p-2) \lambda_{\mathrm{IB}}^{(d-8)/2}\right). \tag{3.3}$$

It is clear that (i) the fluctuation generated second term will dominate for $d < 8$, renormalizing the behaviour of the eigenvalue away from the mean-field value found previously, and (ii) it will still be stable. Although it is tempting to use (3.3) to obtain a self-consistent equation for $\lambda \sim t^{\alpha}$

$$\alpha = \frac{d-8}{2}\alpha + 2 \rightarrow \alpha = \frac{4}{10-d}$$

this approach would fail to consider the full effect of higher-order couplings not included here which renormalize the Gaussian fluctuations eigenvalues in a similar way.

The correct dependence can be obtained from the standard renormalization group approach, through the integration of the differential recursion relation for the quartic coupling (which has now a ω^4 term coming from the diagrams in figures 2 and 3)

$$\frac{\mathrm{d}y}{\mathrm{d}l} = (4-d)y + A\omega^4 \tag{3.4}$$

together with the equation for the cubic coupling, $\mathrm{d}w/\mathrm{d}l = [(6-d)/2]\omega$. These equations have a solution which is dominated far from criticality by $y(l) \simeq \omega^4(l)$ when $6 < d < 8$. Note that A is a positive quantity and all the signs are consistently reversed with respect to the Ising case discussed in [7]. Then we can relate the critical correlation length with its long-l value through $\xi \sim e^{l}\xi(l)$, and at a length scale far away from criticality (determined by $t(l^*) \equiv t\,e^{2l^*} = 1$) we can use the mean-field results obtained in I for the correlation length associated with these modes

$$\xi(l^*) = \left(\frac{y(l^*)t^2(l^*)}{\omega^2(l^*)}\right)^{-1/2}. \tag{3.5}$$

Replacing the various l dependences in (3.5), and that in the expression for the critical ξ, we obtain

$$\xi \sim \exp(l^*)\omega(l^*)^{-1} \sim \exp\left[(l^*)\left(\frac{d-4}{2}\right)\right] = t^{-(d-4)/4}. \tag{3.6}$$

The temperature dependence of the leading behaviour of the eigenvalue changes from the mean-field value t^2 at eight dimensions to a renormalized value $t^{(d-4)/2}$, and at $d=6$ the behaviour $\lambda_{\mathrm{IB}} \sim t$ is obtained. The associated correlation length exponent $\nu_{\mathrm{soft}} = (d-4)/4$ becomes $\frac{1}{2}$ there, as in the case of the non-soft modes.

The fluctuations restore the scaling behaviour of the solution, as far as the IB family of modes is concerned, at $d=6$. A result similar to this was first proposed for Ising spin glasses [7].

4. Renormalization of the intragroup fluctuations

Considering the contributions of all the quartic terms that can be obtained from the third and fourth terms in expression (2.6), the renormalized quartic coupling is

$$3^4 \times 48\omega^4\Delta \sum_{j=0}^{4}\left(\frac{p-2}{12}\right)^j\left(\frac{1}{6}\right)^{4-j}\frac{1}{j!(4-j)!}S_j(R_1, R_2, R_3, R_4)q^{\mathrm{I}}_{R_1}q^{\mathrm{I}}_{R_2}q^{\mathrm{I}}_{R_3}q^{\mathrm{I}}_{R_4} \tag{4.1}$$

where

$$S_j(R_1, R_2, R_3, R_4) = \sum_{R_5, R_6, R_7, R_8} \mathbf{T}(R_1, R_5, R_6)\mathbf{T}(R_2, R_6, R_7)\mathbf{T}(R_3, R_7, R_8)\mathbf{T}(R_4, R_8, R_5)$$

j of the $\mathbf{T}$ factors are cubic replica couplings of the type $\mathbf{T}_A$ introduced in (2.3a), and the remaining $4-j$ belong to the type $\mathbf{T}_B$, defined in (2.3b) and the sum over the modes in the loop extends only over the fast replicon modes to obtain the most diverging contribution, as discussed previously. Δ is the value of the integral over momentum space, introduced in (2.5) and obtained previously.

Although evaluating S_j in general is very difficult [12], it can be obtained for the Gaussian fluctuations case, when two of its variables are mean-field-like longitudinal modes and the corresponding matrices $\mathbf{R}$ in replica space are breathing modes (equation (A2.2a) of I) denoted as $\mathbf{R}_B$.

It is easy to show, in this case, using the orthogonality of the fluctuation modes and the property P1, that

$$\begin{aligned} \mathbf{T}_A(R_B, R_1, R_2) &= \delta_{R_1 R_2} && \text{for any } R_1, R_2 \\ \mathbf{T}_B(R_B, R_1, R_2) &= -\delta_{R_1 R_2} && \text{for } R_1, R_2 \text{ replicon modes.} \end{aligned} \tag{4.2}$$

With this simplification, the expansion around the stable solution can be performed, and the sum over all diagrams, (4.1), reduces to

$$\tfrac{3}{16}\omega^4 q^2 \Delta(p-4)^2\{\mathbf{T}_{AA}[(p-2)/2]^2 + \mathbf{T}_{AB}(p-2) + \mathbf{T}_{BB}\} q^{\mathrm{l}}_{R_1} q^{\mathrm{l}}_{R_2}. \tag{4.3}$$

The couplings $\mathbf{T}_{AA}$, $\mathbf{T}_{AB}$ and $\mathbf{T}_{BB}$, which depend on R_1 and R_2, are defined as

$$\mathbf{T}_{AA}(R_1, R_2) = \sum_{\substack{R,R' \\ \text{replicon modes}}} \mathbf{T}_A(R_1, R, R')\mathbf{T}_A(R_2, R, R') \tag{4.4}$$

with analogous definitions for $\mathbf{T}_{AB}$ and $\mathbf{T}_{BB}$. They are evaluated in appendix 3, where we obtain

$$\mathbf{T}_{AA} = \frac{(m-3)^3 - (m-5)}{2(m-1)(m-2)^2}\delta_{R_1 R_2} \tag{4.5a}$$

$$\mathbf{T}_{AB} = \frac{3m^2 - 15m + 16}{2(m-1)(m-2)^2}\delta_{R_1 R_2} \tag{4.5b}$$

$$\mathbf{T}_{BB} = \frac{m^2(m-5)(m-3) + 4(m^2 - m - 4)}{4(m-1)(m-2)^2}\delta_{R_1 R_2}. \tag{4.5c}$$

With these values, the contribution to the renormalization of the DR replicon mode R_1 in (4.3) can be evaluated close to the continuous transition, where

$$m = \frac{p-2}{2} + \mathrm{O}(t)$$

as obtained in I. The result obtained is

$$\tfrac{3}{128}\omega^4 q^2[(p-4)^3/(p-6)^2](p^3 - 19p^2 + 120p - 236)\Delta(1 + \mathrm{O}(t))(q_{R_1})^2 \tag{4.6}$$

and the corresponding eigenvalue λ_{DR} becomes

$$\lambda_{\mathrm{DR}} = (\lambda_{\mathrm{DR}})_{\text{mean field}} + \tfrac{3}{128}\omega^4[(4-p)^3/(p-6)^2]\Delta P(p) q^2 \tag{4.7}$$

where $P(p)$ is the cubic polynomial obtained in (4.6).

As in the case of the intergroup modes, close to the transition the second term will dominate, as Δ, the momentum integral, diverges below eight dimensions for $t \to 0$. The crucial difference comes from the fact that the polynomial P changes sign for $p = p_1 \simeq 3.77$ and remains negative for $p < p_1$.

For $p > p_1$ the fluctuations renormalize the behaviour of these modes in the same way as the IB modes discussed in section 3. In this range of values of the parameter p, *all* the soft modes are renormalized by the fluctuations in such a way that the scaling behaviour of the solution is restored at the upper critical dimension $d=6$, and all fluctuation modes have correlation lengths that diverge at the transition as $\zeta \sim 1/t^{1/2}$.

For $p_0 < p < p_1$, and in particular for the physical case of the three-state Potts model, the fluctuations destroy the stability of the solution, as the quartic coupling constant flows towards a negative value.

Although in this and the previous section we calculated the one-loop corrections to the eigenvalues associated with the propagating modes via the renormalization of the quartic coupling, in order to emphasize the similarities (and differences) with a similar approach followed in Ising spin glasses, it is not difficult to see that all the higher-order couplings can be renormalized in the same way and the corresponding contributions will be of the same order (in t).

For example, the corrections to the DR eigenvalue (the second term on the RHS of (4.7)) coming from the nth coupling would be

$$\frac{n-1}{2^{n+3}}\,\omega^n \frac{(4-p)^{n-1}}{(p-6)^2}\,\Delta_n P(p) q^{n-2} \rightarrow \frac{n-1}{2^{n+3}}\,\omega^2 \frac{(4-p)}{(p-6)^2}\,P(p) t^{(d-4)/2}$$

where in the last step the value of the integral Δ_n (generalization of (2.5)) and of the mean-field parameter q have been replaced.

In the renormalization group approach leading to the correlation length exponent (3.6) we can use the asymptotic behaviour $y_n(l) \sim \omega^n(l)$, valid for $d>6$, and, as the n-coupling constant appears only in the combination y_n/ω^{n-2}, all the steps leading to (3.6) can be reproduced.

In fact the results of the last two sections can be formally obtained without making any reference to the renormalization of a coupling constant, just by expanding the effective action around the mean-field solution in the ordered phase, and diagonalizing the quadratic part of it to obtain the masses of the eigenmodes to the Gaussian level, as was done in I, and using these as the 'unperturbed' masses of the propagators. The higher-order terms of the action are, then, included in a perturbative approach to renormalize these masses, and, if only the leading one-loop correction is kept (the 'bubble' diagram proportional to ω^2), they will again renormalize only the masses of the soft modes, with a term proportional to $t^{(d-4)/2}$, and there is a strict one-to-one correspondence between these and the diagrams in figures 2 and 3.

A treatment along similar lines was used in the ordered phase of an Ising spin glass [13] keeping only cubic terms in the effective action and integrating the recursion relations, and it was shown there that under the renormalization outlined above the propagator associated with the replicon [14] modes develops an instability. In that case such behaviour could be expected since this mode was found to be unstable already at the Gaussian level [15], at least in the infinite-range model, and the assumed symmetry between replicas has to be broken in order to find a stable solution.

It is interesting that a generalization of that approach to the case of the Potts glass will also show, in spite of the stability of the solution at the Gaussian level, the presence of an instability for a certain range of values of p, as discussed before, signalling the presence of some new behaviour, which was not present in the Gaussian analysis. Although the situation found here, as mentioned earlier, has its similarities with scenarios associated with fluctuation-driven first-order transitions, in the sense that the contributions coming from the fluctuations renormalize the couplings out of the region of stability in parameter space, there are several caveats that have to be made.

(i) The full renormalization group equations for this system (including up to the quartic couplings) have not been solved (even at the one-loop level) and therefore we cannot track exactly the flow of the couplings, which would be necessary to investigate the existence of a first-order transition.

(ii) There are contributions to the fluctuations coming from diagrams with more than one loop which become important at critical dimensions higher than six and were not considered here, and which may lead to some cancellations. This is believed to happen, for example, in the Parisi solution of the Ising spin glass [4]. The whole loop expansion would have to be considered in order to make more definite predictions.

(iii) Even at a more basic level it is possible that the existence of the instability might be signalling the fact that we are performing an expansion around the 'wrong' solution in replica space, and that our starting replica symmetry breaking ansatz (equation (4.1) of I) has to be abandoned. In this connection it should be remembered that, as shown in I, we cannot obtain a new solution by carrying the replica symmetry breaking scheme one (or more) step further in the Parisi hierarchy [16], at the mean-field level, because we regain the same solution [17]. In the next section we will use the full spectrum of fluctuations evaluated earlier to calculate the one-loop corrections to the free energy and the equation of state. It would be interesting to investigate if the new free energy obtained in this way is also stable against further replica symmetry breaking. This implies the calculation of corrections to the eigenvalues obtained in I (which enter in the expression for the fluctuation corrected free energy (5.2)) by the terms generated by the extra symmetry breaking steps. Such a calculation is presently in progress. If the system becomes unstable against further replica symmetry breaking (as it has been found to happen at the mean-field level in this model at lower temperatures [18] this could provide a way out of the paradox implied by the instability found for $p < p_1$†.

5. Corrections to the equation of state

The corrections to the mean-field free energy due to the fluctuations can be obtained by realizing that if we expand the effective Hamiltonian of the system (equation (2.13) of I) to second order in the fluctuations $R_{ab}^{\alpha\beta}$ ($Q_{ab}^{\alpha\beta}(x) \equiv q_{\mathrm{MF}} + R_{ab}^{\alpha\beta}(x)$), and we transform the integration variables in the functional integral of Z to the eigenmodes of the problem found before, the partition function becomes, in momentum space

$$Z = \int \prod_i \mathrm{d}R_i(k) \exp\left(-H_{\mathrm{MF}} - \sum_i \int (k^2 + \lambda_i) R_i^2(k)\right) \tag{5.1}$$

where the λ_i are the eigenvalues found in I. Then, performing the simple Gaussian integrals, the free energy becomes

$$\frac{F}{kT} = H_{\mathrm{MF}} + \frac{1}{2}\sum_i \int \mathrm{d}^d k \ln(k^2 + \lambda_i) \tag{5.2}$$

where unimportant constants have been omitted. This is exactly equivalent to summing the contributions of all one-loop diagrams in the loop expansion [11].

The momentum integral can be performed exactly in any (real) dimension d and expanding to cubic order in the eigenvalues, the fluctuation term becomes

$$\frac{1}{\Gamma(d/2)(4\pi)^{d/2}} \sum_i \left(\frac{\lambda_i}{d-2}\Lambda^{d-2} - \frac{\lambda_i^2}{2(d-4)}\Lambda^{d-4} + \frac{\lambda_i^3}{3(d-6)}\Lambda^{d-6}\right)$$

† This possibility was suggested by C de Dominicis.

where Λ is a cut-off associated with the discreteness of the lattice. We have explicitly used the fact that $d>6$ in order to neglect terms of the type $\lambda_i^{d/2}$ and $\lambda_i^{d/2}\ln\lambda_i$.

Using the expression for the eigenvalues $\lambda_i=-t_0+a_iq+b_iq^2+c_iq^3+\dots$ and the explicit value of the cut-off (2π in our units) for an hypercubic lattice, this can be expressed as

$$\frac{\pi^{d/2-6}}{\Gamma(d/2)}\left\{\frac{q^2}{4}\left[\frac{\pi^4}{d-2}B-\frac{\pi^2}{8(d-4)}A_2+\frac{t}{4}\left(\frac{\pi^2}{d-4}B-\frac{1}{4(d-6)}A_2\right)\right]\right.$$
$$\left.+\tfrac{1}{6}q^3\left(\frac{3}{2}\frac{\pi^4}{d-2}C-\frac{3\pi^2}{8(d-4)^4}AB+\frac{1}{32(d-6)}A_3\right)\right\} \tag{5.3}$$

where we introduced the constants

$$B=\sum_i b_i \qquad C=\sum_i c_i \qquad A_2=\sum_i a_i^2 \qquad A_3=\sum_i a_i^3 \qquad AB=\sum_i a_ib_i \tag{5.4}$$

and used the fact that $A=\Sigma_i a_i=0$, as can be verified using A3.3–A3.16 of I.

The sums over the eigenvalues defined in (5.4) are evaluated in appendix 4. If we insert in (5.3) the values obtained there, and we add the mean-field free energy obtained previously (equation (4.2) of I) the total free energy can be expressed as

$$\frac{F}{nkT}=(m-1)(p-1)\left\{-\tfrac{1}{4}tq^2-\tfrac{1}{6}q^3\left[\left((m-2)+\frac{p-2}{2}\right)\mu\right.\right.$$
$$\left.\left.-\frac{a}{d-2}\left((3p-4)(m-2)-\frac{p-2}{2}(p-3)^2\right)\right]\right\} \tag{5.5}$$

where the constants a and μ are defined as

$$a=\frac{\pi^{d/2-6}}{2^5\Gamma(d/2)} \qquad \mu=1-48a\pi^2\left(\frac{1}{d-2}-\frac{p^2-8p+6}{8(d-4)}\right).$$

The parameter μ depends only very mildly on the value of p, in the range of validity of this solution and can be, for the purpose of this calculation, replaced by its value at $p=4$. The transition temperature, as expected, is reduced by the fluctuations from its original value of $T_{G_0}=J$ to

$$T_G\simeq J\left[1-a\left(\frac{(2\pi)^4}{d-2}-\frac{2\pi^2(p^2-8p+8)}{d-4}\right)\right].$$

Again, this quantity depends very weakly on p, and the reduction is of 16% at $d=8$, and 20% at $d=7$. The new reduced temperature t is defined as $t=(T_G-T)/J$. The saddle point equations associated with the free energy (5.5) can be obtained in analogy with the procedure followed in I (section 4) and the solutions obtained (compare with expressions (4.5) obtained in I), are

$$q=0 \qquad t<0$$
$$q=\frac{t}{(4-p)\mu-[a/(d-6)][6p-8+(p-2)(p-3)^2]}+O(t^2) \qquad t>0 \tag{5.6a}$$

$$m=\frac{p-2}{2}\frac{(d-6)+(a/\mu)(p-3)^2}{(d-6)-(a/\mu)(3p-4)}+O(t). \tag{5.6b}$$

In the $d\to\infty$ limit the mean-field solution is recovered, and when we approach $d=6$

the deviations from it become more important. (Of course, the equations are not valid in the 'neighbourhood' of $d = 6$, where the logarithmic corrections we discarded to arrive at the free energy have to be included.)

The interesting feature here is that p_c, the critical value of p for which the solution ceases to exist (which is 4 at the mean-field level), diminishes when the dimensionality is decreased; and, to the level of the approximation made, it is still valid that the value of p at which the continuous solution ceases to exist (where the denominator of (5.6a) vanishes, and the cubic term of F changes sign) corresponds to the value at which m reaches the endpoint of its physical range of variation, $m(t=0) = 1$

$$p_c = 2 + \frac{2(d-6)}{(d-6) + (a/\mu)[(p-3)^2 + 3p - 4]} \approx 4 - \frac{18}{d-6}\left(\frac{a}{\mu}\right).$$

In the last step we used the fact that (a/μ) is a small parameter, $a/\mu \simeq 10^{-3}$ for $d \leqslant 8$.

Although the correction is small unless we are close to the critical dimension, it is interesting that it goes in the direction of narrowing the interval of values of p for which a continuous stable solution exists, as the instability found in the last section did.

Complete knowledge of the fluctuation spectrum would permit as, at least in theory, to tackle the problem of the free energy of the system in $d < 6$, where the long-range fluctuations dominate; but the peculiar situation found here, where an instability appears at even higher dimension, requires us to clarify this situation first, to find the 'correct' minimum in replica space to describe the system. In this sense, the calculation in this section should be viewed simply as a first step, to obtain a new free energy functional which can be used to explore the existence of other ground states.

As mentioned before, at least one natural step would be to verify if the feature present in the mean-field theory of the Potts glasses and other spin glasses (see discussion in I), namely that no further replica symmetry breaking is allowed, will still be preserved when the new free energy functional is considered.

The main result of this work is to show that the effects of fluctuation around the mean-field theory of a spin glass can be studied. Furthermore this knowledge can be used, although so far only in a crude way, to test the theories themselves.

Important progress has also been made recently in the simulation of Potts glasses in three dimensions [19, 20] and some hope exists of even exploring higher dimensions in order to make connections with field theoretical efforts.

In conclusion, we summarize the main results obtained here.

We considered a p-state Potts spin glass, for which a stable mean-field theory phase transition had been found before, and using the knowledge of the complete set of eigenmodes of fluctuations around it, we showed that the presence of soft modes requires the renormalization of some of the couplings already at $d = 8$. We evaluated the contribution of the fluctuations (to one-loop) to the renormalization and found that while one of the modes is renormalized in a conventional way, in the sense that the fluctuations, when included, restore its scaling behaviour at $d = 6$, there is a replicon mode which behaves quite differently. The fluctuations generated by this mode renormalize the coupling in such a way that, for certain values of p (in particular for the three-state Potts model), it moves away from the region of stability and its mass becomes negative. It is not clear at this point if this effect signals a true instability which will be present at all orders in the loop expansion, or whether some cancellations may occur at higher orders. Finally, for $d > 6$, corrections to the free energy due to the fluctuations were obtained, and a dimensional dependence of the boundary between first- and second-order transitions was found.

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Appendix 1

In the mean-field solution with one level of RSB around which we are expanding, the only eigenmodes which are different from zero in the ordered phase are the members of the DB family (diagonal in the Potts indices and independent of replica indices, as long as both of them belong to the same group).

When expanding around this mean-field solution, quartic couplings generated by diagrams where more than two of the vertices are of the type represented in figures 1(a) and 1(b), where *all* the arms represent IB modes, will not contribute to the Gaussian fluctuation eigenvalues, as they are of higher order in the fluctuations.

Also, diagrams with only one vertex of the type in figure 1(b) are clearly impossible if the condition $\mathrm{I} \neq \mathrm{J} \neq \mathrm{K}$ has to be satisfied among its group indices. So the only possible diagram containing these vertices is the one depicted in figure 2(a), and its permutations.

The diagram with one vertex of the type shown in figure 1(a) and the rest of the type shown in figure 1(d) does not contribute to the expansion around the mean field, a result which follows easily using (A1.1) of I, as they include at least one Potts tensor v_{abc} with self-contracted indices.

Finally, diagrams with four vertices of the type shown in figure 1(d) will contribute only to the renormalization of the quadratic (non-leading) part of the non-soft longitudinal DB mode, and are, therefore, not relevant here.

Appendix 2

As we want the contribution of the diagrams in figure 2(a) and 2(b) to the Gaussian eigenvalues, the external single line modes have to be 'longitudinal' (mean-field-like) modes of the family DB. The corresponding vertices, the last term in (2.6), then become

$$-\omega \frac{(m-1)}{2} q \sum_{\mathrm{I} \neq \mathrm{J}} (q_{ab}^{\mathrm{IJ}})^2. \tag{A2.1}$$

Using this expression in evaluating the diagrams, the diagonal character of the coupling follows immediately, and the expression

$$\left(\frac{\omega(m-1)}{2} q\right)^2 \left[c_1 \left(\frac{\omega}{6}\right)^2 \left(\frac{n}{m} - 2\right)(p-1) + c_2 \left(\frac{\omega}{12m}\right)^2 (p-2)^2\right] \Delta \tag{A2.2}$$

is obtained, where the c_i are the combinatoric coefficients, the factor $[(n/m)-2]\,(p-1)$ reflects the sum over the internal loop in figure 2(a), and the contraction of the Potts tensors, together with (A1.4) of I, gives the factor $(p-2)^2$ in the contribution from figure 2(b). Δ is the integral

$$(2\pi)^{-d}\int \frac{d^d k}{(k^2+\lambda_{IB})^4}.$$

Inserting the combinatorial factors and the value of the integral [11], we obtain (3.1).

Appendix 3

Using the definitions (2.3a) and (2.3b) of the replica couplings, we obtain the following expressions:

$$\mathbf{T}_{AA}(R_1, R_2) = \sum_{\alpha\neq\beta}\sum_{\gamma\neq\delta}(R_1)_{\alpha\beta}(R_2)_{\gamma\delta}(\mathbf{S}_{\alpha\beta,\gamma\delta})^2 \tag{A3.1}$$

$$\mathbf{T}_{AB}(R_1, R_2) = \sum_{\alpha\neq\beta}\sum_{\gamma\neq\delta}(R_1)_{\alpha\beta}(R_2)_{\gamma\delta}\sum_{\varepsilon\neq\gamma,\delta}\mathbf{S}_{\alpha\beta,\gamma\varepsilon}\mathbf{S}_{\alpha\beta,\delta\varepsilon} \tag{A3.2}$$

$$\mathbf{T}_{BB}(R_1, R_2) = \sum_{\alpha\neq\beta}\sum_{\gamma\neq\delta}(R_1)_{\alpha\beta}(R_2)_{\gamma\delta}\sum_{\substack{\theta\neq\alpha,\beta\\ \varepsilon\neq\gamma,\delta}}\mathbf{S}_{\alpha\theta,\gamma\varepsilon}\mathbf{S}_{\beta\theta,\delta\varepsilon} \tag{A3.3}$$

where we introduced the tensor

$$\mathbf{S}_{\alpha\beta,\gamma\delta} = \sum_{R=\text{Replicon}}(R)_{\alpha\beta}(R)_{\gamma\delta} \tag{A3.4}$$

and the sum over R is restricted only to the replicon modes obtained in (A2.2c) of I, so we cannot use the completeness of the matrices $\mathbf{R}$. This last tensor can be evaluated using the explicit expression of the orthogonal linear combinations of those modes [12]. Alternatively, it can be obtained by realizing that, by symmetry, the tensor takes only three different values

$$x = \mathbf{S}_{\alpha\beta,\alpha\beta}$$

$$y = \mathbf{S}_{\alpha\beta,\alpha\gamma} \qquad \gamma\neq\beta$$

$$z = \mathbf{S}_{\alpha\beta,\gamma\delta} \qquad \gamma\neq\alpha,\ \beta,\ \delta\neq\alpha,\ \beta.$$

Using the definition (A3.4) and the property P1, it easily follows that

$$\sum_{\alpha}\mathbf{S}_{\alpha\beta,\gamma\delta} = 0 \Rightarrow \begin{cases} x+(m-2)y=0 \\ 2y+(m-3)z=0 \end{cases} \tag{A3.5}$$

and, from the orthonormality of the fluctuation modes,

$$\sum_{\alpha\neq\beta}\mathbf{S}_{\alpha\beta,\alpha\beta} \equiv m(m-1)x$$

$$= \text{number of independent replicon modes} = \frac{m(m-3)}{2}. \tag{A3.6}$$

From (A3.5) and (A3.6) it follows immediately that

$$\mathbf{S}_{\alpha\beta,\alpha\beta} \equiv x = \frac{m-3}{2(m-1)}$$

$$\mathbf{S}_{\alpha\beta,\alpha\gamma} \equiv y = -\frac{(m-3)}{2(m-1)(m-2)} \tag{A3.7}$$

$$\mathbf{S}_{\alpha\beta,\gamma\delta} \equiv z = \frac{1}{(m-1)(m-2)}.$$

Using these values for the tensor $\mathbf{S}_{\alpha\beta,\gamma\delta}$ in (A3.1), we obtain

$$\mathbf{T}_{AA}(R_1, R_2) = 2x^2 \sum_{\alpha\neq\beta} (R_1)_{\alpha\beta}(R_2)_{\alpha\beta} + 4y^2 \sum_{\alpha\neq\beta\neq\gamma\neq\alpha} (R_1)_{\alpha\beta}(R_2)_{\alpha\gamma}$$
$$+ z^2 \sum_{\substack{\alpha,\beta,\gamma,\delta \\ \text{all different}}} (R_1)_{\alpha\beta}(R_2)_{\gamma\delta}$$
$$= (2x^2 - 4y^2 + 2z^2)\delta_{R_1R_2}$$

where in the last step the orthogonality of the replicon modes and the property P1 was used repeatedly. In the same way, some tedious but straightforward algebra in (A3.2) and (A3.3) gives

$$\mathbf{T}_{AB}(R_1, R_2) = [2my^2 + 2(m-4)z^2 - 4xy - 4(m-3)yz]\delta_{R_1R_2}$$
$$\mathbf{T}_{BB}(R_1, R_2) = [(m-2)x^2 + (m-4)^2y^2 + (m^2 - 9m + 22)(z^2 - 2yz)$$
$$- 2(m-2)xy + 4xz]\delta_{R_1R_2}.$$

Inserting the values of x, y and z obtained in (A3.7) in these expressions, we obtain the results (4.5a–c).

Appendix 4

The calculation of the sums of the different pieces of the eigenvalues can be performed using their explicit expressions given in I.

It is easier, however, to obtain them directly from the expression of the fluctuation matrix (equations (3.6), (5.1) of I and the fifth-order term proportional to q^3 not given before, which we write here for completeness

$$\mathbf{M}_3^1 = \delta_{ac}\delta_{bd}\{\mathbf{A}[(m-2)(m-3+2(p-2)) + \tfrac{1}{6}(p^2 - 6p + 6)]$$
$$- \mathbf{B}[(p-2)(2m-9) + \tfrac{1}{6}(p^2 - 6p + 6) + (m-9)(m-3)] - \mathbf{C}(p + 2m - 14)\}$$
$$- (F_{abcd}/p)[\![\mathbf{A}\{(m-2)[(m-3) + 2(p-2)] + \tfrac{1}{6}(p^2 - 6p + 6)\}$$
$$+ \mathbf{B}(7m + 4p - 30) + 8\mathbf{C}]\!]$$
$$+ \delta_{ab}\delta_{cd}\{[p - 2 + 2(m-2)](\mathbf{A} + \mathbf{B}) + \mathbf{C}\} \qquad (\text{A4.1}a)$$
$$\mathbf{M}_3^2 = -\delta_{ac}\delta_{bd}\{\mathbf{B}[\tfrac{1}{6}(p^2 - 6p + 6) + 2(m-2)(p-2)] + \mathbf{C}[(p-2) + 2(m-2)]\} \qquad (\text{A4.1}b)$$

where the notation of the replica matrices was introduced in (3.7) of I.

Let us call the parts of the fluctuation matrix proportional to q, q^2 and q^3, respectively $\mathbf{M}_1$, $\mathbf{M}_2$ and $\mathbf{M}_3$.

The quantities B, C, A_2, A_3 and AB can be expressed as $\operatorname{Tr}\mathbf{M}_2$, $\operatorname{Tr}\mathbf{M}_3$, $\operatorname{Tr}(\mathbf{M}_2)^2$, $\operatorname{Tr}(\mathbf{M}_2)^3$ and $\operatorname{Tr}(\mathbf{M}_1 \cdot \mathbf{M}_2)$.

Furthermore, the block structure of the matrix induced by the replica symmetry breaking scheme implies that

$$\operatorname{Tr}(\mathbf{M}_i)^r = \frac{n}{m}\operatorname{Tr}(\mathbf{M}_i^1)^r + \frac{1}{2}\left(\frac{n}{m} - 1\right)\frac{n}{m}\operatorname{Tr}(\mathbf{M}_i^2)^r$$

where $\mathbf{M}_i^1$ and $\mathbf{M}_i^2$ are the intragroup and intergroup parts of each of the matrices.

Using these two facts, and the expressions for the matrices, the traces in the replica space and the Potts space can be obtained, through some tedious but straightforward algebra. (Although expressions like (A4.1) look quite formidable, the calculations, at least in the linear case, are greatly simplified from the fact that **B** and **C** are traceless.)

The results obtained are listed below:

$$A=\sum_i a_i=0 \qquad B=\frac{c_0}{2} \qquad C=c_0\left(m-2+\frac{p-2}{2}\right)$$

$$A_2=c_0\left((p-1)(n-2)+\frac{(p-2)^2}{2}\right)\to c_0\frac{p^2-8p+8}{2}$$

$$A_3=c_0\left((3p-4)(m-2)-\frac{p-2}{2}(p-3)^2\right)$$

$$AB=c_0\left(m-2+\frac{p-2}{2}\right)[p^2-4p+2+2(n-2)(p-1)]\to c_0\left(m-2+\frac{p-2}{2}\right)\frac{p^2-8p+6}{2}$$

with $c_0=n(m-1)(p-1)$, and in the last step the $n\to 0$ limit was taken.

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